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THEORY OF TORIC VARIETIES FROM THE TOPOLOGICAL VIEWPOINT

Osaka City University Mikiya Masuda (枡田幹也)

1. Introduction

Theory of toric varieties, which was laid down by Demazure, Miyake-Oda and Mumford etc. around 1970, can be viewed as a bridge between algebraic geometry and combinatorics. In fact, it is based on the following two fundamental facts.

1st fundamental fact. There is a one-to-one correspondence between toric varieties of complex dimension n and fans of real dimension n.

2nd fundamental fact. To an ample line bundle over a compact toric variety M of complex dimension n, there is a map (called a moment map) $\Phi\colon M \to \mathbb{R}^n$ such that $\Phi(M)$ is a lattice convex polytope of dimension n.

Here a *toric variety* of complex dimension n is a normal algebraic variety with $(\mathbb{C}^*)^n$-action having a dense orbit, and a *fan* of real dimension n is a collection of cones in $\mathbb{R}^n$ with apex at the origin satisfying certain conditions. The details can be found in [3] and we will illustrate these notions with examples later. Because of the 1st fundamental fact, all geometrical information on a toric variety such as cohomology and characteristic numbers can be read from the associated fan. However, those are topological invariants, so one can expect to develop the theory of toric varieties using only topological technique to some extent. It turns out that this is possible and this point of view enables us to treat more general combinatorial objects what we call *multi-fan* and *multi-polytope*. In this note, we will overview the theory of toric varieties from the topological point of view, which is developed in [7], [11], [8], [9], [13] and [12]. The subjects treated in [2] and [1] are closely related to the subject treated in this note.

2. How to associate a fan with a smooth toric variety

We begin with two simple but basic examples of toric varieties.

Example 2.1. Let $(g_1, \dots, g_n) \in (\mathbb{C}^*)^n$.

(1) $\mathbb{C}^n$ with the action of $(\mathbb{C}^*)^n$ given by

$$(z_1, \dots, z_n) \mapsto (g_1 z_1, \dots, g_n z_n)$$

is a non-compact smooth toric variety.

(2) $\mathbb{C}P^n$ with the action of $(\mathbb{C}^*)^n$ given by

$$[z_1, \dots, z_n, z_{n+1}] \mapsto [g_1 z_1, \dots, g_n z_n, z_{n+1}]$$

is a compact toric variety.

Products of toric varieties are again toric varieties. Rather non-trivial examples of toric varieties are Hirzebruch surfaces.

If M is a toric variety of complex dimension n, then M has a dense orbit by definition, which is isomorphic to $(\mathbb{C}^*)^n$, and other orbits are known to be finitely many and isomorphic to $(\mathbb{C}^*)^k$ for some $k < n$. Let M_i $(i = 1, \dots, d)$ be the closure of complex codimension one orbits. They are invariant divisors and fixed by certain $\mathbb{C}^*$-subgroups. For instance, when $M = \mathbb{C}^n$ in the above example, $d = n$ and M_i's are coordinate hyperplanes $z_i = 0$, and when $M = \mathbb{C}P^n$, $d = n+1$ and M_i's are hypersurfaces $z_i = 0$. The $\mathbb{C}^*$-subgroup which fixes M_i depends on M_i. When M is smooth, M_i is a connected complex codimension one smooth submanifold fixed by a certain $\mathbb{C}^*$-subgroup. M_i's are invariant dovisors and we call M_i a *characteristic submanifold*. We note that $M \setminus \cup_i M_i$ is the dense orbit isomorphic to $(\mathbb{C}^*)^n$. Therefore, one can expect that toric varieties can be determined by characteristic submanifolds and their neighborhoods. We extract two data from them.

1st data. We set

$$\Sigma_M := \{I \subset \{1, \dots, d\} \mid \bigcap_{i \in I} M_i \neq \emptyset\}.$$

One easily checks that this is an abstract simplicial complex. The dimension of the simplicial complex is at most $n-1$ and it attains $n-1$ if and only if the action of $(\mathbb{C}^*)^n$ on M has a fixed point .

2nd data. The set $\mathrm{Hom}(\mathbb{C}^*, (\mathbb{C}^*)^n)$ consisting of homomorphisms from $\mathbb{C}^*$ to $(\mathbb{C}^*)^n$ is an abelian group under the multiplication on $(\mathbb{C}^*)^n$ and naturally isomorphic to $\mathbb{Z}^n$. In the following we make the following identification

$$\mathrm{Hom}(\mathbb{C}^*, (\mathbb{C}^*)^n) = \mathbb{Z}^n.$$

An element in $\mathrm{Hom}(\mathbb{C}^*, (\mathbb{C}^*)^n)$ determines a $\mathbb{C}^*$-subgroup of $(\mathbb{C}^*)^n$ as the image of $\mathbb{C}^*$, and two non-trivial elements in $\mathrm{Hom}(\mathbb{C}^*, (\mathbb{C}^*)^n)$ determine the same $\mathbb{C}^*$-subgroup if and only if their corresponding vectors in $\mathbb{Z}^n$ lie on a same line. To each characteristic submanifold M_i, an element $v_i \in \mathrm{Hom}(\mathbb{C}^*, (\mathbb{C}^*)^n)$ satisfying the following two conditions is uniquely detemined:

(1) $v_i(\mathbb{C}^*)$ fixes M_i
(2) $v_i(g)_* w = gw$ for w in the normal bundle of M_i

where $v_i(g)_*$ denotes the differential of $v_i(g)$ which is a diffeomorphism of M, and gw denotes the complex multiplication of w by $g \in \mathbb{C}^*$ (this meakes sense because the normal bundle of M_i is naturally a complex vector bundle). For each $I \in \Sigma_M$, we span a cone in $\mathrm{Hom}(\mathbb{C}^*, (\mathbb{C}^*)^n) \otimes \mathbb{R} = \mathbb{R}^n$ by v_i's ($i \in I$). The collection of these cones is the fan Δ_M associated with M. This is not a standard way to define a fan. See [3] for the details.

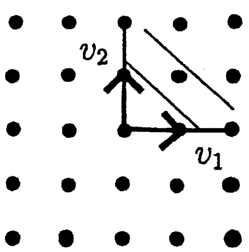
Example 2.2. Take $\mathbb{C}^2$ with the natural action of $(\mathbb{C}^*)^2$ given by

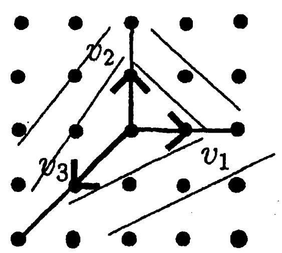
$$(z_1, z_2) \to (g_1 z_1, g_2 z_2).$$

Then $\mathbb{C}^2{}_i := \{z_i = 0\}$ $(i = 1, 2)$ are characteristic submanifolds and

$$\Sigma_{\mathbb{C}^2} = \{\{1\}, \{2\}, \{1,2\}\}$$
$$v_1(g) = (g, 1), \quad v_2(g) = (1, g).$$

Therefore, the fan $\Delta_{\mathbb{C}^2}$ of $\mathbb{C}^2$ with the above action can be described as the left figure below.





Example 2.3. Take $\mathbb{C}P^2$ with the natural action of $(\mathbb{C}^*)^2$ given by

$$[z_1, z_2, z_3] \to [g_1 z_1, g_2 z_2, z_3].$$

Then $\mathbb{C}P^2{}_i := \{z_i = 0\}$ $(i = 1, 2, 3)$ are characteristic submanifolds and

$$\Sigma_{\mathbb{C}P^2} = \{\{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{3,1\}\}$$
$$v_1(g) = (g, 1), \quad v_2(g) = (1, g), \quad v_3(g) = (g^{-1}, g^{-1})$$

Therefore, the fan $\Delta_{\mathbb{C}P^2}$ of $\mathbb{C}P^2$ with the above action can be described as the right figure above.

3. Equivariant cohomology

When a smooth toric variety M is compact, the two data Σ_M and $\{v_i\}$ forming the fan of M have nice interpretation in terms of equivariant cohomology. Let me explain this in this section.

The equivariant cohomology of M with the action of $(\mathbb{C}^*)^n$ is defined by

$$H^*_{(\mathbb{C}^*)^n}(M) := H^*(E(\mathbb{C}^*)^n \times_{(\mathbb{C}^*)^n} M)$$

Let M_i $(i = 1, \dots, d)$ be characteristic submanifolds of M as before. Since M_i is of real codimension two in M, the Poincaré dual of M_i in the equivariant cohomology defines a cohomological degree two element τ_i in $H^2_{(\mathbb{C}^*)^n}(M)$. Intersections of characteristic submanifolds M_i's are transversal, so the Poincaré dual of the intersection $\bigcap_{i\in I} M_i$ for $I \subset \{1, \dots, d\}$ is $\prod_{i\in I} \tau_i \in H^*_{(\mathbb{C}^*)^n}(M)$. This shows that $\prod_{i\in I} \tau_i$ vanishes when $\bigcap_{i\in I} M_i$ is empty. It turns out that $H^*_{(\mathbb{C}^*)^n}(M)$ is generated by τ_i's as a ring and the relations are generated by these monomials.

Theorem 3.1 (Well-known). *As a ring*

$$H^*_{(\mathbb{C}^*)^n}(M) = \mathbb{Z}[\tau_1, \dots, \tau_d] \Big/ \Big(\prod_{i\in I} \tau_i \mid \bigcap_{i\in I} M_i = \emptyset \Big)$$

In the terminology of commutative algebra, one can say that $H^*_{(\mathbb{C}^*)^n}(M)$ as a ring is isomorphic to the face ring of the simplicial complex Σ_M.

Example 3.2. $H^*_{(\mathbb{C}^*)^2}(\mathbb{C}P^2) = \mathbb{Z}[\tau_1, \tau_2, \tau_3]/(\tau_1\tau_2\tau_3)$

In short, we might say that

$$\boxed{H^*_{(\mathbb{C}^*)^n}(M) \textit{ as a ring} \iff \text{the simplicial complex } \Sigma_M \text{ of } M.}$$

However, the equivariant cohomology has a finer structure than the ring structure. We have a fibration

$$M \longrightarrow E(\mathbb{C}^*)^n \times_{(\mathbb{C}^*)^n} M \xrightarrow{\pi} B(\mathbb{C}^*)^n \tag{3.1}$$

Through $\pi^*\colon H^*(B(\mathbb{C}^*)^n) \to H^*_{(\mathbb{C}^*)^n}(M)$, one can view $H^*_{(\mathbb{C}^*)^n}(M)$ as an algebra over $H^*(B(\mathbb{C}^*)^n)$. As is well-known, $H^*(B(\mathbb{C}^*)^n)$ is a polynomial ring generated by its degree two part. Therefore, the algebra structure can be determined by the image of elements in $H^2(B(\mathbb{C}^*)^n)$ by π^*.

Lemma 3.3. *To each $i \in \{1, \dots, d\}$, there is a unique element $v_i \in H_2(B(\mathbb{C}^*)^n)$ such that*

$$\pi^*(u) = \sum_{i=1}^{d} \langle u, v_i \rangle \tau_i \quad \textit{for } \forall u \in H^2(B(\mathbb{C}^*)^n)$$

Moreover, viewing v_i as an element of $\operatorname{Hom}(\mathbb{C}^*, (\mathbb{C}^*)^n)$ *through an identification*

$$H_2(B(\mathbb{C}^*)^n) = [B\mathbb{C}^*, B(\mathbb{C}^*)^n] = \operatorname{Hom}(\mathbb{C}^*, (\mathbb{C}^*)^n),$$

$v_i(\mathbb{C}^*)$ is the $\mathbb{C}^*$-subgroup of $(\mathbb{C}^*)^n$ which fixes M_i.

In short, we might say that

$$\boxed{H^*_{(\mathbb{C}^*)^n}(M) \text{ as an } \textit{algebra} \text{ over } H^*(B(\mathbb{C}^*)^n) \iff \text{the fan } \Delta_M \text{ of } M}$$

Remark. It follows from the fibration (3.1) that $\pi^*(H^{>0}(B(\mathbb{C}^*)^n))$ maps to zero in the ordinary cohomology $H^*(M)$ by the restriction map from $H^*_{(\mathbb{C}^*)^n}(M)$ to $H^*(M)$. This is equivalent to saying that $\pi^*(H^2(B(\mathbb{C}^*)^n))$ maps to zero in $H^*(M)$ by the restriction map because $H^*(B(\mathbb{C}^*)^n)$ is generated by its degree two part. It turns out that the restriction map is surjective and its kernel is generated by elements in $\pi^*(H^2(B(\mathbb{C}^*)^n))$. Thus we have

$$H^*(M) = H^*_{(\mathbb{C}^*)^n}(M)/\big(\pi^*(u) \mid u \in H^2(B(\mathbb{C}^*)^n)\big)$$

Combining this with Theorem 3.1 and Lemma 3.3, we obtain a well-known explicit description of $H^*(M)$ as a ring, see [3, p.106].

4. Torus manifolds and multi-fans

In order to associated a fan with a smooth toric variety M, we needed two data, the simplicial complex Σ_M and integral vectors v_i's in $\operatorname{Hom}(\mathbb{C}^*, (\mathbb{C}^*)^n) = H_2(B(\mathbb{C}^*)^n)$. We note that

$$\operatorname{Hom}(\mathbb{C}^*, (\mathbb{C}^*)^n) = \operatorname{Hom}(S^1, T) = H_2(BT),$$

where $T = (S^1)^n$, so those two data can be defined even if we restrict the $(\mathbb{C}^*)^n$-action to T. This observation leads us to study T-actions. The reader should notice that the argument developed in Section 3 also works for the restricted T-action. In the following we use $H_2(BT)$ instead of $\operatorname{Hom}(S^1, T)$.

Let M be an orientable closed smooth manifold of $\dim_{\mathbb{R}} = 2n$ with smooth T-action such that $M^T \neq \emptyset$. Note that

$$\dim T = \frac{1}{2}\dim M.$$

Let M_i $(i = 1, \dots, d)$ be a connected closed real codimension two submanifold fixed by a certain S^1-subgroup of T. We call these M_i's characteristic submanifolds of M as before.

Definition. When M and M_i's are oriented, we call M a *torus mani-*

Compact smooth toric varieties with restricted T-actions provide examples of torus manifolds. However, there are many torus manifolds which are not toric varieties. We shall give two such examples. The reader can find more examples in [2], [11], [1].

Example 4.1 (Torus manifolds which are not toric).

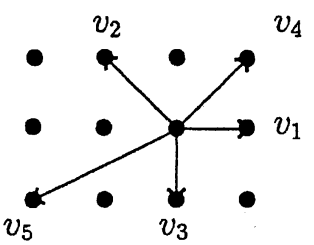
(1) The unit sphere S^{2n} of $\mathbb{C}^n \times \mathbb{R}$ with a natural T-action defined by
$$(z_1, \ldots, z_n, y) \to (g_1 z_1, \ldots . g_n z_n, y)$$
is a torus manifold but this is not a toric variety when $n \geq 2$.

(2) Let N be a smooth manifold of dimension n with boundary diffeomorphic to S^{n-1} and let M be a compact smooth toric manifold of complex dimension n (with the restricted T-action). We remove an open invariant tubular neighborhood of a free orbit from M, and paste it with $N \times T$ equivariantly along their boundary. The resulting space $M(N)$ is a torus manifold with orbit space diffeomorphic to N because the orbit space of a compact smooth toric variety with the restriction T-action is an n-ball. Therefore, $M(N)$ is not a toric variety unless N is a ball.

For a torus manifold M, a simplicial complex Σ_M of dimension $n-1$ and $v_i \in H_2(BT) = \mathbb{Z}^n$ can be defined, and one can form cones in $H_2(BT;\mathbb{R}) = \mathbb{R}^n$ similarly to the toric case. However, unlike the toric case, these cones may overlap as shown in the following example.

Example 4.2 (see [11]). Let $v_1, \ldots, v_d$ $(d \geq 3)$ be a sequence of vectors in $\mathbb{Z}^2$ such that each successive pair v_i and v_{i+1} is a basis of $\mathbb{Z}^2$ for $i \in \{1, \ldots, d\}$ where $v_{d+1} = v_1$ (see the figure below). Then there is a torus manifold M of (real) dimension 4 having d number of characteristic submanifolds M_i $(i = 1, \ldots, d)$ such that

(1) the associated simplicial complex is
$$\Sigma_M = \{\{1\}, \ldots, \{d\}, \{1,2\}, \ldots, \{d-1, d\}, \{d, 1\}\}$$
(2) the vector in $H_2(BT) = \mathbb{Z}^2$ corresponding to M_i is the given v_i.



In the case shown in the figure above, any generic point of $H_2(BT;\mathbb{R})$ is contained in exactly two cones. This degree of overlap of cones is roughly speaking equal to the Todd genus of M. Let me state this more precisely.

Let I be a subset of $\{1, \dots, d\}$ with cardinality n such that $M_I := \cap_{i \in I} M_i$ is non-empty. Then the intersection consists of finitely many T-fixed points. When M is a toric variety, the intersection consists of exactly one point if it is non-empty. But this is not the case for a torus manifold. For instance, when $M = S^{2n}$ in Example 4.1, the intersection of n characteristic submanifolds consists of two points (the north pole and the south pole) for $n \geq 2$. Let $p \in M_I$. Then the tangential T-module $\tau_p M$ at p decomposes into

$$\tau_p M = \bigoplus_{i \in I} \tau_p M / \tau_p M_i. \tag{4.1}$$

Here each factor has an orientation since M and M_i are both oriented by the definition of torus manifold, and it induces an orientation on the right-hand side of (4.1) which is independent of the order of the sum because $\tau_p M / \tau_p M_i$ is real 2-dimensional. This orientation may not agree with the given orientation on $\tau_p M$ (the left-hand side of (4.1)). We give $+1$ or -1 to the T-fixed point p according as the two orientations at (4.1) agree or disagree. We count the number of points in M_I with this sign and denote its sum by $w(I)$. When M is a toric variety, $w(I) = 1$ for all I. When $M = S^{2n}$ for $n \geq 2$, I is unique (since $d = n$) and $w(I) = 0$. We denote by $\angle v_I$ the cone (of dimension n) spanned by v_i's for $i \in I$.

Theorem 4.3 ([11]). *Let v be a generic element of $H_2(BT; \mathbb{R})$. Then $\sum_{v \in \angle v_I} w(I)$ is independent of the choice of v and this integer agrees with the "Todd genus" of M.*

Remark. (1) The author does not know whether an arbitrary torus manifold admits a T-invariant unitary (or weakly almost complex) structure compatible with the orientations on M and M_i's. Neverthless, "Todd genus" can be defined for a torus manifold using Lefschtez fixed point formula (see [8]) in such a way that it agrees with the Todd genus of M when M admits a unitary structure.

(2) The Todd genus of a compact smooth toric variety is one. This together with the above theorem explains why cones have no overlap in an ordinary fan.

For a torus manifold M, the collection of cones formed from the two data Σ_M and v_i's together with the weight function w on cones of maximal dimension n is called the *multi-fan* of M. When M is toric, the weight function is constant and takes the value one as remarked above. Therefore, the multi-fan of M can be viewed as the ordinaly fan of M when M is toric.

We have a correspondence

$$\Psi \colon \{\text{Torus manifolds}\} \to \{\text{multi-fans}\}$$

but this is NOT one-to-one. For instance, the torus manifolds $M(N)$ in Example 4.1 (2) have the same multi-fan as the toric variety M regardless of N.

Problem. Characterize the image and "kernel" of Ψ. (See [8] for some work on this problem.)

Although the map Ψ is not injective, characteristic numbers such as Euler characteristic, T_y-genus and elliptic genus can be described explicitly in terms of multi-fans ([11], [8], [9]).

5. Moment maps and polytopes

We shall discuss about the 2nd fundamental fact mentioned in the Introduction. First we note

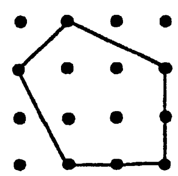
$$\mathrm{Lie}(T)^* = H^2(BT;\mathbb{R}) \supset H^2(BT)$$

Fans or multi-fans are defined in the *homology* $H_2(BT;\mathbb{R})$ while moment maps have images in the *cohomology* $H^2(BT;\mathbb{R})$.

To an *ample* T-line bundle $L \to M$ over a *compact toric variety* M, there exists a moment map

$$\Phi\colon M \to \mathrm{Lie}(T)^* = H^2(BT;\mathbb{R})$$

and $\Phi(M)$ is a *lattice* convex polytope, where lattice polytope means that the vertices of the polytope lie on the lattice $H^2(BT) = \mathbb{Z}^n$ as shown in the following figure.



There is a natural identification

$$H^2(BT) = \mathrm{Hom}(T, S^1),$$

so a lattice point in $H^2(BT)$ can be interpreted as a complex one-dimensional T-module. We denote by t^u the complex T-module corresponding to $u \in H^2(BT)$.

Theorem 5.1 (Well-known). *Let $L \to M$ be an ample line bundle over a compact smooth toric variety. Then*

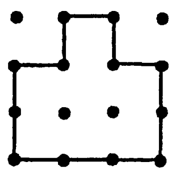
$$H^0(M;L) = \sum_{u\in\Phi(M)\cap H^2(BT)} t^u \qquad \textit{as complex } T\textit{-modules}$$

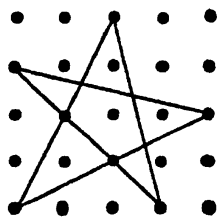
Since the line bundle L is ample, its higher dimensional cohomology groups $H^i(M;L)$ $(i>0)$ vanish. Therefore, the left-hand side at the identity of the theorem above may be viewed as the equivariant Riemann-Roch number $RR^T(M;L)$ of the line bundle L. Thus, forgetting the action, the identity in the theorem above reduces to

$$RR(M;L) = \sharp(\Phi(M)) \tag{5.1}$$

where $RR(M,L)$ is the non-equivariant Riemann-Roch number of L and $\sharp(\Phi(M))$ denotes the number of lattice points in $\Phi(M)$.

To an *arbitrary* T-line bundle $L \to M$ over a *torus manifold* M, the equivariant Riemann-Roch number $RR^T(M,L)$ and the moment map Φ are still defined. But $RR^T(M;L)$ may be a virtual T-module, i.e., an element of the representation ring $R(T)$ of T and $\Phi(M)$ is not necessarily a convex polytope as shown in the following figures. The latter leads to the notion of *multi-polytope*, see [8].





Theorem 5.2 ([10], [5], [11]).

$$RR^T(M,L) = \sum_{u\in\Phi(M)} m(u)t^u \in R(T)$$

and $m(u) \in \mathbb{Z}$ can be described in terms of $\Phi(M)$.

When $\Phi(M)$ is the figure shown above, $m(u)$ is roughly speaking the rotation number of the boundary of $\Phi(M)$ around u. For example, if u is inside (resp. outside) of the left figure above, then $m(u) = \pm 1$ (resp. $m(u) = 0$). If u is inside of the pentagon of the right (star-shaped) figure above, then $m(u) = \pm 2$.

The theorem above implies that (5.1) would still hold once we define the right-hand side $\sharp(\Phi(M))$ in an appropriate way. In fact, this is done in [8] and some results on counting lattice points on lattice *convex* polytopes such as Pick's formula, Ehrhart polynomial and Khovanskii-Pukhlikov formula are generalized to lattice *multi-polytopes* with suitable modification ([11], [8]).

6. Orbit spaces

In this section we discuss about the orbit space M/T of a torus manifold M. First we note that the orbit space of $\mathbb{C}^n$ by the restricted standard T-action in Example 2.1 (1) is naturally identified with the first quadrant $(\mathbb{R}_{\geq 0})^n$ of $\mathbb{R}^n$. In fact, the map $(z_1, \ldots, z_n) \to (|z_1|, \ldots, |z_n|)$ induces the identification. We say that a torus manifold M is *locally standard* if every point in M has an invariant neighborhood U weakly equivariantly diffeomorphic to an open invariant subset W of $\mathbb{C}^n$ with the restricted standard T-action. Here "weakly equivariantly diffeomorphic" means that there is an automorphism $\psi \colon T \to T$ and a diffeomorphism $f \colon U \to W$ such that $f(ty) = \psi(t)f(y)$ for all $t \in T, y \in U$. Since the orbit space $\mathbb{C}^n/T = (\mathbb{R}_{\geq 0})^n$ is a manifold with corners, the following lemma is immediate from the definition of locally standardness.

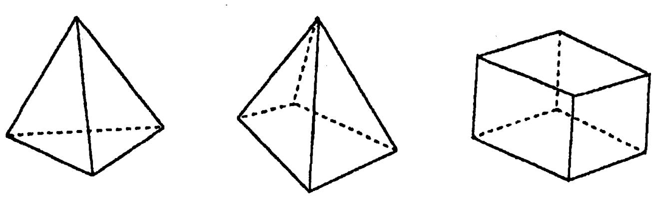
Lemma 6.1. *If a torus manfold M is locally standard, then the orbit space M/T is a manifold with corners.*

Here is a sufficient condition for a torus manifold to be locally standard.

Theorem 6.2 ([13]). *A torus manifold M is locally standard if $H^{odd}(M) = 0$.*

When M is locally standard, possible isotropy groups are subtorus of T and if H is a subtorus of dimension k, then every connected component in the H-fixed point set M^H is of codimension $2k$. The image of a connected component in M^H by the quotient map $M \to M/T$ is called a *face* of codimension k in M/T and a codimension one face is called a *facet*.

A compact smooth toric variety M with the restricted T-action is locally standard, so its orbit space is a manifold with corners. If M is projective in addition (hence M admits an ample T-line bundle), then the moment map $\Phi \colon M \to H^2(BT; \mathbb{R})$, which is equivariant and the T-action on the taget space $H^2(BT; \mathbb{R})$ is trivial, induces an idenfication between the orbit space M/T and the *simple* convex polytope $\Phi(M)$. Here a convex polytope of dimension n is said to be simple if there are exactly n edges meeting at every vertex. For example, the middle polytope in the following figure is not simple while the others are simple.



simple not simple simple

Here are some explicit examples of orbit spaces.

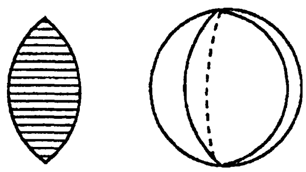
Example 6.3. (1) $(\mathbb{C}P^1)^n/T = (\mathbb{C}P^1/S^1)^n = [0,1]^n$ (n-cube)
(2) $\mathbb{C}P^n/T =$ n-simplex
In fact, the map from $\mathbb{C}P^n$ to $\mathbb{R}^n$ given by

$$[z_1, \ldots, z_n, z_{n+1}] \to \frac{1}{\sum_{i=1}^{n+1} |z_i|^2}(|z_1|^2, \ldots, |z_n|^2)$$

induces an identification between $\mathbb{C}P^n/T$ and the standard n-simplex.

Example 6.4. The torus manifolds in Example 4.1 are locally standard but their orbit spaces are not simple convex polytopes.

(1) The orbit space of S^{2n} by the restricted T-action is homeomorphic to an n-ball. As a manifold with corners, it has two vertices corresponding to the north and south poles and has n facets as shown in the following figure.



S^4/T^2 S^6/T^3

(2) As remarked in Example 4.1, the orbit space $M(N)/T$ can be identified with N which is not necessarily acyclic.

It is well-known that the cohomology ring of a compact smooth toric variety is generated by its degree two part (see the Remark at the end of Section 3). It is also known that every face in the orbit space of a compact smooth toric variety by the restricted T-action is contractible (in particular, acyclic) and any multiple intersection of faces is connected (unless it is non-empty) like a simple convex polytope.

Theorem 6.5 ([13]). *Let M be a torus manifold.*

(1) *$H^{odd}(M) = 0$ if and only if M is locally standard and M/T is face-acyclic (i.e., every face in M/T is acyclic).*
(2) *$H^*(M)$ is generated by its degree two part if and only if M is locally standard and M/T is a homology polytope (i.e., face-acyclic and any multiple intersection of faces is connected unless it is non-empty).*

Remark. If we cut the face-acyclic manifold S^{2n}/T along a vertex, then it turns into an n-simplex which is the orbit space $\mathbb{C}P^n/T$, observe this with Example 6.4 (1). In general, given a face-acyclic manifold Q which may have disconnected intersections of faces, one can convert Q into a homology polytope P by cutting along faces in Q. When Q is the orbit space of a torus manifold M, cutting along a face F in Q corresponds to blowing up M along the preimage $q^{-1}(F)$ where $q\colon M \to$

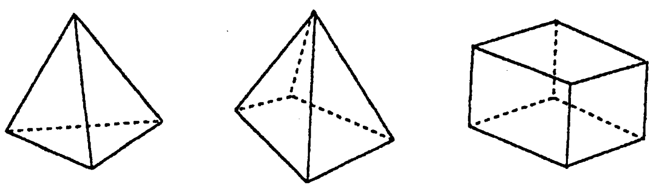
$Q = M/T$ is the quotient map. This together with the theorem above implies that one can convert a torus manifold M with $H^{odd}(M) = 0$ into a torus manifold whose cohomology is generated by its degree two part by blowing up M finitely many times along submanifolds fixed by subtorus.

7. Face numbers

Let P be a (not necessarily simple) convex polytope of dimension n. For $i = 0, 1, \dots, n$ we set

$$f_{i-1} := \text{ the number of codimension } i \text{ faces of } P.$$

We note that $f_{-1} = 1$, f_0 is the number of facets and f_{n-1} is the number of vertices. The vector $(f_0, f_1, \dots, f_{n-1})$ is called the f-vector of P.



$(f_0, f_1, f_2) = (4, 6, 4)$ $(5, 8, 5)$ $(6, 12, 8)$

The f-vector of a convex polytope P cannot be arbitrary. In fact,

$$\sum_{i=0}^{n}(-1)^{n-i} f_{i-1} = 1 \tag{7.1}$$

because the left-hand side must agree with the Euler characteristic of the convex polytope which is one. When P is simple, further equalities are known among components f_i's. In order to state this, it is convenient to introduce the h-vector of P defined from the equation

$$\sum_{i=0}^{n} h_i t^{n-i} := \sum_{i=0}^{n} f_{i-1}(t-1)^{n-i}$$

Needless to say, the h-vector has the same information as the f-vector. Check that in the above polytopes

$$(h_0, h_1, h_2, h_3) = (1, 1, 1, 1) \quad (1, 2, 1, 1) \quad (1, 3, 3, 1)$$

We note that $h_0 = 1$ and $h_n = \sum_{i=0}^{n}(-1)^{n-i} f_{i-1}$, so the equation (7.1) can be restated as $h_n = h_0$.

Theorem 7.1 (Well-known as Dehn-Sommerville equations). *If a convex polytope P is simple, then $h_i = h_{n-i}$ for any i.*

A simple convex polytope is a face-acyclic manifold with corners and f- and h-vectors can be defined for a manifold with corners in a similar fashion.

Theorem 7.2. *Dehn-Sommerville equations $h_i = h_{n-i}$ still hold for a face-acyclic manifold Q with corners.*

Outline of proof. One can find a torus manifold (or orbifold) M such that $M/T = Q$ and $h_i = \operatorname{rank} H^{2i}(M)$. Then Poincaré duality on M implies $h_i = h_{n-i}$. □

If P is a simple convex polytope, then there is a compact toric *orbifold* M with $M/T = P$ and it is well-known that the h-vector of P must satisfy not only Dehn-Sommerville equations but also some inequalities obtained by applying the hard Lefschetz theorem to M. The h-vectors of simple convex polytopes are characterized and known as g-theorem, see [3]. The following theorem characterizes h-vectors of face-acyclic manifolds with corners.

Theorem 7.3 ([12]). *A vector of integers $(h_0, \dots, h_n)$ with $h_0 = h_n = 1$ is an h-vector of a face-acyclic manifold Q (of dimension n) with corners if and only if*

(1) *$h_i = h_{n-i}$ for any i,*
(2) *$h_i \geq 0$ for any i, and*
(3) *$\sum_{j=0}^{n} h_j$ is even if $h_j = 0$ for some $j \geq 1$.*

Idea of proof. The "if" part is easy. As for the "only if" part, the condition (1) is Theorem 7.2 and the condition (2) follows from the fact that $h_i = \operatorname{rank} H^{2i}(M)$ in the proof of Theorem 7.2. The difficult part is the condition (3) conjectured by Stanley [14]. The idea to deduce the condition (3) is as follows. Suppose we find a torus manifold M such that $M/T = Q$ and $H^{odd}(M) = 0$. (Unfortunately, this is not always possible although M can be taken as a torus *orbifold*.) Then the total Stiefel-Whitney class of M is of the form

$$w(M) = \prod_{i=1}^{d} (1 + \mu_i) \in H^*(M; \mathbb{Z}/2)$$

where $\mu_i \in H^2(M; \mathbb{Z}/2)$ is the Poincaré dual of a characteristic submanifold M_i $(i = 1, \dots, d)$. If $h_j = 0$ for some $j \geq 1$, then $H^{2j}(M; \mathbb{Z}/2) = 0$ and hence the top Stiefel-Whitney class $w^{2n}(M)$ vanishes. Therefore, $\sum_i h_i = \chi(M)$ must be even. □

It follows from Theorem 7.3 that $(1, 0, 2, 0, 1)$ does appear as an h-vector of a face-acyclic manifold with corners but $(1, 0, 1, 0, 1)$ does not.

8. Equivariant cohomology of a torus manifold

The equivariant cohomology $H^*_T(M)$ of a torus manifold M has an explicit description when $H^{odd}(M) = 0$, which generalizes Theorem 3.1. For a face F of M/T, let M_F be the inverse image of F by the quotient map $M \to M/T$ and let $\tau_F \in H^{2\operatorname{codim} F}_T(M)$ be the Poincaré dual of M_F in equivariant cohomology.

Theorem 8.1 ([13]). *If M is a torus manifold with $H^{odd}(M) = 0$,*

$$H^*_T(M) = \mathbb{Z}[\tau_F \mid F \text{ face}]\Big/\Big(\tau_G\tau_H - \tau_{G\vee H}\sum_{E\in G\cap H}\tau_E\Big)$$

where $G \vee H$ is the minimal face of M/T which contains G and H.

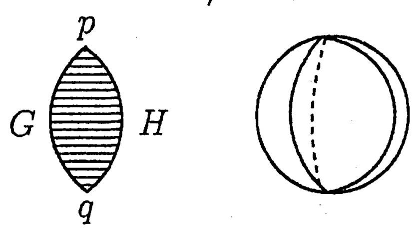
A key fact used to prove this theorem is that if $H^{odd}(M) = 0$, then the restriction map $H^*_T(M) \to H^*_T(M^T)$ is injective and its image is determined by the one-skeleton of M/T (see [4], [6]).

When $H^*(M)$ is generated by its degree two part (this is the case when M is a compact smooth toric variety), M/T is a homology polytope by Theorem 6.5; so any multiple intersection of faces is connected (if it is non-empty) and any face can be obtained as a multiple intersection of facets. Therefore, the right-hand side at the identity in Theorem 8.1 reduces to that in Theorem 3.1 when $H^*(M)$ is generated by its degree two part.

We may view Theorem 8.1 through the dual of M/T. When M/T is a homology polytope (e.g., a simple convex polytope), its dual is the simplicial complex Σ_M. However, the dual of M/T is not necessarily a simplcial complex in general. For instance, the dual of S^n/T ($n \geq 2$) is two $(n-1)$-simplices glued together along their boundary. (Observe this for the figure below.) In general, the dual of a manifold with corners is what we call a *simplicial cell complex* ([13]) or a *simplcial poset* ([14]), and Theorem 8.1 says that $H^*_T(M)$ is the face ring of the simplcial cell complex (or simplicial poset) associated with M. When all intersections of faces of M/T are connected, the associated simplicial cell complex agrees with the simplicial complex Σ_M.

Example 8.2. We have

$$H^*_T(S^4) = \mathbb{Z}[\tau_G, \tau_H, \tau_p, \tau_q]\Big/\big(\tau_G\tau_H - (\tau_p + \tau_q)\big)$$



S^4/T^2 $\qquad$ S^6/T^3

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DEPARTMENT OF MATHEMATICS, OSAKA CITY UNIVERSITY, SUGIMOTO, SUMIYOSHI-KU, OSAKA 558-8585, JAPAN

E-mail address: masuda@sci.osaka-cu.ac.jp